Hypercyclicity of composition operators in pseudoconvex domains

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Abstract

We characterize hypercyclic composition operators $C_\varphi : f \mapsto f \circ \varphi$ on the space of functions holomorphic on Ω, where $\Omega \subset \mathbb{C}^N$ is a pseudoconvex domain and φ is a holomorphic self-mapping of Ω.

In the case when all the balls with respect to the Carathéodory pseudodistance are relatively compact in Ω, we show that much simpler characterisation is possible (many natural classes of domains satisfy this conditioon, i.e. strictly pseudoconvex domains, bounded convex domains, etc.).

Moreover, we show that in such a class of domains, and in simply connected or infinitely connected planar domains, hypercyclicity of C_φ implies its hereditary hypercyclicity.

1 Introduction

Let Ω be a pseudoconvex domain in $\mathbb{C}^N$ and let $\varphi : \Omega \to \Omega$ be a holomorphic mapping. We are interested in the problem of hypercyclicity and hereditary hypercyclicity of the composition operator $C_\varphi : f \mapsto f \circ \varphi$ on the space $\mathcal{O}(\Omega)$ of holomorphic functions $f : \Omega \to \mathbb{C}$, endowed with the usual topology of locally uniform convergence.

In the case when Ω is a planar domain, a complete characterization of hypercyclicity was given by Grosse-Erdmann and Mortini in [6]. In higher dimensions the problem was considered by several authors, mostly in cases when Ω is a polydisc, an euclidean ball or the whole $\mathbb{C}^N$ with φ being special (see [2] and the references in [6]). Analogous problem in spaces of real analytic functions was considered in [3] and in kernels of general (non-necessarily Cauchy-Riemann) partial differential equations in [10].

In this paper we give a characterization of hypercyclicity and hereditary hypercyclicity of C_φ for arbitrary pseudoconvex domain $\Omega \in \mathbb{C}^N$ and arbitrary holomorphic mapping $\varphi : \Omega \to \Omega$ (Theorem 3.4), using ideas developed by several authors. It is formulated in the language of holomorphic hulls of compact subsets of Ω, which in dimension one coincides with the notion of Ω-convexity used in [6]. However, our best results are contained in Sections 5, 6 and 7.

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A disadvantage of the conditions in Theorem 3.4 is that they require to answer the question: when a sum of two disjoint holomorphically convex sets are holomorphically convex? On the other hand, the necessary conditions (Proposition 3.1), which avoid this question, turn out to be sufficient in the cases of simply connected and infinitely connected planar domains (see [6, Theorem 3.21]). This fact motivated us to ask about domains in $\mathbb{C}^N$ having the same property. We deal with this topic in Section 5, showing that it holds for Ω if the closed balls with respect do the Carathéodory pseudodistance are compact in the topology of Ω (Theorem 5.4). Such a class of domains includes many 'nice' domains, e.g. bounded convex domains, strictly pseudoconvex domains, analytic polyhedra, etc. (see Corollary 5.5).

As a simple observation it follows from Theorem 3.4 that the operator C_φ is hypercyclic if and only if it is hereditarily hypercyclic with respect to some sequence $(n_l)_l$ (Observation 3.6). As we show in section 6, in many domains even more is true: C_φ is hypercyclic if and only if it is hereditarily hypercyclic (Theorem 6.2). Using this result, we conclude that domains with compact Carathéodory balls have also this property. As the second conclusion, in Section 7 we show that hypercyclicity and hereditary hypercyclicity are equivalent in simply connected and infinitely connected planar domains, what is an improvement of Theorem 3.21 in [6]. It remains an open problem if every hypercyclic operator C_φ is automatically hereditarily hypercyclic.

Given an increasing sequence $(n_l)_l \in \mathbb{N}$, in this paper we study the following problems:

(p1) Hypercyclicity of C_φ.

(p2) Hypercyclicity of C_φ with respect to $(n_l)_l$.

(p3) Hereditary hypercyclicity of C_φ with respect to $(n_l)_l$.

(p4) Hereditary hypercyclicity of C_φ.

For the definition of this notions see Definition 2.1. It is clear that there hold the implications (p4) $\Rightarrow$ (p3) $\Rightarrow$ (p2) $\Rightarrow$ (p1).

2 Preliminaries

For an open set $\Omega \in \mathbb{C}^N$, by $\mathcal{O}(\Omega, \mathbb{C}^M)$ be denote the space of all holomorphic functions $f : \Omega \to \mathbb{C}^M$, equipped with the compact-open topology, i.e. the topology of uniform convergence on compact subsets. In the case $M = 1$ we shortly write $\mathcal{O}(\Omega)$ instead of $\mathcal{O}(\Omega, \mathbb{C})$. For an open set $\Omega' \subset \mathbb{C}^M$ by $\mathcal{O}(\Omega, \Omega')$ we denote the set of all holomorphic mappings from Ω to Ω'; this set is open in $\mathcal{O}(\Omega, \mathbb{C}^M)$.

For a domain $\Omega \subset \mathbb{C}^N$ by $\widetilde{\Omega} = \Omega \cup \{\infty_\Omega\}$ we denote the usual compactification of Ω by an element $\infty_\Omega \notin \Omega$.

We denote: $\mathbb{D}$ - the unit disc in $\mathbb{C}$, $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$ and $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$.

For a boundary point $a \in \partial\Omega$ we call a holomorphic function $F : \Omega \to \mathbb{D}$ a *peak function* for a if $\lim_{\Omega \ni z \to a} F(z) = 1$. Although usually a peak function is defined in a different way and requires stronger conditions, the above definition is sufficient for our considerations.

For a compact set $K \subset \Omega$ by $\widehat{K}_\Omega$ or $(K)\widehat{\ }_\Omega$ we denote the *holomorphic hull* of the set K with respect to Ω, i.e.

$$\widehat{K}_\Omega := \{z \in \Omega : |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\}.$$

The set K is called *holomorphically convex* if $K = \widehat{K}_\Omega$; we call such set shortly: Ω*-convex*. For $\Omega = \mathbb{C}^N$ we shortly write $\widehat{K}$ and $(K)\widehat{\ }$. To get more informations, see [7].

We say that a sequence $(K_l)_l$ of compact sets is an *exhaustion* of Ω if $\bigcup_l K_l = \Omega$ and $K_l \subset \operatorname{int} K_{l+1}$.

For pseudoconvex domains $U \subset \Omega \subset \mathbb{C}^N$ we say that U is a Runge domain with respect to Ω if every function from $\mathcal{O}(U)$ can be approximated locally uniformly on U by functions from $\mathcal{O}(\Omega)$. For some important facts connected with this notion see [7].

We say that a sequence of holomorphic functions $f_n : \Omega \to \Omega'$ is *compactly divergent* (*in* $\mathcal{O}(\Omega, \Omega')$) if for each compact subsets $K \subset \Omega$, $L \subset \Omega'$ there is n_0 such that $f_n(K) \cap L = \varnothing$ for all $n \geq n_0$. We say that$(f_n)_n$ is *run-away* (*in* $\mathcal{O}(\Omega, \Omega')$) if for each compact subsets $K \subset \Omega, L \subset \Omega'$ there is n such that $f_n(K) \cap L = \varnothing$. In the case $\Omega = \Omega'$ it is always enough to consider the situation when $L = K$. Note that the sequence $(f_n)_n$ is run-away if and only if it has a compactly divergent subsequence, and $(f_n)_n$ is compactly divergent if and only if each of its subsequences is run-away.

For a domain $\Omega \subset \mathbb{C}^N$ and points $z, w \in \Omega$ let

$$\begin{aligned} c_\Omega^*(z, w) &:= \sup\{|F(z)| : F \in \mathcal{O}(\Omega, \mathbb{D}), F(w) = 0\}, \\ c_\Omega(z, w) &:= \log \frac{1 + c_\Omega^*(z, w)}{1 - c_\Omega^*(z, w)}. \end{aligned}$$

Here c_Ω is the Carathéodory pseudodistance, and c_Ω^* is the Möbius pseudodistance in Ω. For more informations, we refer the reader to [9].

Given a set X, a mapping $T : X \to X$ and an integer number n, we denote by $T^{[n]}$ the n-th iteration of T, i.e. the mapping $T \circ T \circ \ldots \circ T$ (n times).

Definition 2.1. Let T and T_n ($n \in \mathbb{N}$) be continuous self-mappings of a topological space X.

1. We call a point $x \in X$ an *universal element* for $(T_n)_n$, if the set $\{T_n(x) : n \in \mathbb{N}\}$ is dense in X.

2. We say that the sequence $(T_n)_n$ is *universal* if there exists a universal element for it, and we call this sequence *hereditarily universal* if each of its subsequences is universal.

3. We say that the mapping T is *hypercyclic* (resp. *hereditarily hypercyclic*) with respect to an increasing sequence $(n_l)_l \subset \mathbb{N}$ if the sequence $(T^{[n_l]})_{n_l}$ is universal (resp. hereditarily universal).

4. We say that the sequence $(T_n)_n$ is *topologically transitive* if for every non-empty open subsets $U, V \subset X$ there exists n such that $T_n(U) \cap V \neq \varnothing$.

Below we recall two theorems which are essential for our considerations (see [5, Theorem 1 and Proposition 1]).

Theorem 2.2. *Let X be a separable Fréchet space. A sequence $(T_n)_n$ of continuous self-maps of X is topologically transitive if and only if the set of its universal elements is dense in X.*

Moreover, if one of these conditions holds, then the set of universal elements for $(T_n)_n$ is a dense G_δ-subset of X.

Theorem 2.3. *Let X be a separable Fréchet space. Suppose that $(T_n)_n$ is a sequence of continuous self-maps of X such that each T_n has dense range and that the family $(T_n)_n$ is commuting, i.e.*

$$T_n \circ T_m = T_m \circ T_n, \quad \text{for } m, n \in \mathbb{N}.$$

Then the set of universal elements for $(T_n)_n$ is empty or dense.

From now we consider the case when X is the space $\mathcal{O}(\Omega)$ and T is the composition operator $C_\varphi : f \mapsto f \circ \varphi$ for a pseudoconvex domain $\Omega \subset \mathbb{C}^N$ and a holomorphic mapping $\varphi : \Omega \to \Omega$.

As mentioned, we are interested in problems (p1)-(p4). It is clear that (p1) (resp. (p4)) is just (p2) (resp. (p3)) with $n_l = l$. Also observe that if C_φ is hypercyclic, i.e. the condition (p1) is fulfilled, then for any $n \in \mathbb{N}$ the map ${C_\varphi}^{[n]}$ has dense range. Indeed, if f is a universal element for $({C_\varphi}^{[n]})_n$, then $f \circ \varphi^{[k]} = {C_\varphi}^{[n]}(f \circ \varphi^{[k-n]})$ belongs to the range of ${C_\varphi}^{[n]}$ for each $k \geq n$. It is obvious that the sequence $({C_\varphi}^{[n_l]})_l$ is commuting, so as a corollary from Theorems 2.2 and 2.3, we obtain the following fact:

Corollary 2.4. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \subset \mathbb{N}$ be an increasing sequence. Then the following conditions are equivalent:*

(1) There exists an universal element for $({C_\varphi}^{[n_l]})_l$.

(2) The set of universal elements of $({C_\varphi}^{[n_l]})_l$ is a dense G_δ-subset of $\mathcal{O}(\Omega)$.

(3) The sequence $({C_\varphi}^{[n_l]})_l$ is topologically transitive.

Since the sets

$$W_{f_0,K,\epsilon} := \{f \in \mathcal{O}(\Omega) : |f - f_0| < \epsilon \text{ on } K\}, \ f_0 \in \mathcal{O}(\Omega), \epsilon > 0, K \subset \Omega \text{ compact},$$

form a basis of the topology of $\mathcal{O}(\Omega)$, the topological transitivity of a subsequence $({C_\varphi}^{[n_l]})_l$ means that for every $\epsilon > 0$, g, h holomorphic on Ω and compact $K \subset \Omega$ there are $l \in \mathbb{N}$ and a function f holomorphic on Ω such that

$$|f - g| < \epsilon \text{ on } K \text{ and } |f \circ \varphi^{[n_l]} - h| < \epsilon \text{ on } K.$$

Assuming that the mapping φ is injective (as we shall see in Proposition 3.1, it is necessary even for hypercyclicity of C_φ), the above condition takes the form:

$$\text{(tt)} \qquad |f - g| < \epsilon \text{ on } K \text{ and } |f - h \circ \varphi^{[-n_l]}| < \epsilon \text{ on } \varphi^{[n_l]}(K).$$

We are going to formulate our theorems in the language of holomorphic hulls of compact subsets of Ω. By this reason, let us recall some well-known properties of holomorphic hulls:

Lemma 2.5. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain and let $K, L \subset \Omega$ be compact and such that $\widehat{K}_\Omega \cap \widehat{L}_\Omega = \varnothing$. Then the following conditions are equivalent:*

(1) $\widehat{K \cup L}_\Omega = \widehat{K}_\Omega \cup \widehat{L}_\Omega$.

(2) There exist open and disjoint subsets $U, V \subset \Omega$ such that $\widehat{K}_\Omega \subset U$, $\widehat{L}_\Omega \subset V$ and $\widehat{K \cup L}_\Omega \subset U \cup V$.

(3) There exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $\widehat{F(K)} \cap \widehat{F(L)} = \varnothing$.

For the reader's convenience, we present a sketch of the proof.

Sketch of the proof. (1) $\Rightarrow$ (3): Consider the function f equal 0 in a neighborhood of $\widehat{K}_\Omega$ and 1 in a neighborhood of $\widehat{L}_\Omega$. By (1), this function is holomorphic in a neighborhood of the Ω-convex set $\widehat{K \cup L}_\Omega$, so it can be approximated on this set by functions holomorphic on Ω (see [7, Theorems 4.3.2 and 4.3.4]). Hence there is some $F \in \mathcal{O}(\Omega)$ such that $|F - f| < \frac{1}{2}$ on $\widehat{K \cup L}_\Omega$. This F satisfies (3).

(3) $\Rightarrow$ (2): It suffices to define $U := F^{-1}(U_0), V := F^{-1}(V_0)$, where U_0 and V_0 are some disjoint open neighborhoods of the compact sets $\widehat{F(K)}$ and $\widehat{F(L)}$, respectively.

(2) $\Rightarrow$ (1): The right-to-left inclusion is obvious, so we prove the other one. Fix $z_0 \in I := \widehat{K \cup L}_\Omega$. We can assume that $z_0 \in U$. We prove that $z_0 \in \widehat{K}_\Omega$. The characteristic function χ_U of U, restricted to the set $U \cup V$, is (by (2)) holomorphic in a neighborhood of the Ω-convex set I, so there exists a sequence of functions $(g_n)_n \subset \mathcal{O}(\Omega)$ uniformly convergent to χ_U on I. Hence $g_n \to 1$ on $I \cap U$ and $g_n \to 0$ on $I \cap V$. For any function $f \in \mathcal{O}(\Omega)$ the sequence $(fg_n)_n$ converges uniformly to f on $I \cap U$ and to 0 on $I \cap V$, so

$$\begin{aligned} |f(z_0)| &= \lim_{n\to\infty} |f(z_0)g_n(z_0)| \le \lim_{n\to\infty} \sup_{z\in K\cup L} |f(z)g_n(z)| \\ &= \lim_{n\to\infty} \sup_{z\in K} |f(z)g_n(z)| = \sup_{z\in K} |f(z)|. \end{aligned}$$

This implies that $z_0 \in \widehat{K}_\Omega$ and finishes the proof. ☐

Lemma 2.6. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain and let $K, L \subset \Omega$ be compact subsets of Ω. If $K \cap L = \varnothing$ and the set $K \cup L$ is Ω-convex, then K and L are both Ω-convex.*

Proof. As above, we can find a function $F \in \mathcal{O}(\Omega)$ such that $|F| < \frac{1}{2}$ on K and $|F - 1| < \frac{1}{2}$ on L. There is $\widehat{K}_\Omega \subset K \cup L$ and

$$\widehat{K}_\Omega \cap L \subset F^{-1}\left(\widehat{F(K)}\right) \cap F^{-1}(f(L)) \subset F^{-1}\left(\frac{1}{2}\mathbb{D}\right) \cap F^{-1}\left(1 + \frac{1}{2}\mathbb{D}\right) = \varnothing,$$

so $\widehat{K}_\Omega \subset K$. ☐

3 General results

We start this section with formulating some necessary conditions. In fact, they appear in several papers.

Proposition 3.1. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Suppose that C_φ is hypercyclic w.r.t. $(n_l)_l$. Then:*

(c1) The mapping φ is injective.

(c2) The image $\varphi(\Omega)$ is a Runge domain w.r.t. Ω.

(c3) The sequence $(\varphi^{[n_l]})_l$ is run-away.

Remark 3.2. Note that the first condition implies that φ is a biholomorphism on its image, which is a classical result ([8, Theorem 2.2.1]). Also note that the second condition means that the set $\varphi(K)$ is Ω-convex for each compact Ω-convex subset $K \subset \Omega$ ([7, Theorem 4.3.3]). This immediately implies that for any integer number n the set $\varphi^{[n]}(K)$ also is Ω-convex.

Proof of Proposition 3.1. Let f be a universal function for $({C_\varphi}^{[n_l]})_l$. The first part is obvious. For the second we need to prove that the restrictions $g|_{\varphi(\Omega)}, g \in \mathcal{O}(\Omega)$, are dense in $\mathcal{O}(\varphi(\Omega))$. Let $h \in \mathcal{O}(\varphi(\Omega))$. Then $h \circ \varphi$ is holomorphic on Ω, so there is a sequence $(l_k)_k$ such that $f \circ \varphi^{[n_{l_k}]} \to h \circ \varphi$ on Ω. Hence $f \circ \varphi^{[n_{l_k}-1]} \to h$ on $\varphi(\Omega)$, as the mapping φ is a biholomorphism on its image.

We prove the third part. Let $K \subset \Omega$ be compact. For each $j \in \mathbb{N}$ there exists l_j such that $|f \circ \varphi^{[n_{l_j}]} - j| \leq \frac{1}{j}$ on K. This implies

$$\inf_{z \in \varphi^{[n_{l_j}]}(K)} |f(z)| = \inf_{z \in K} |f \circ \varphi^{[n_{l_j}]}(z)| \geq j - \frac{1}{j} > \sup_{z \in K} |f(z)| \quad \text{for big } j,$$

so $\varphi^{[n_{l_j}]}(K) \cap K = \varnothing$. ☐

Necessary conditions for hereditary universality are given by the following twin proposition:

Proposition 3.3. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Suppose that C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$. Then:*

(h1) The mapping φ is injective.

(h2) The image $\varphi(\Omega)$ is a Runge domain w.r.t. Ω.

(h3) The sequence $(\varphi^{[n_l]})_l$ is compactly divergent.

Proof. The first two parts follow from the previous proposition. For the last one it is enough to recall that a sequence of holomorphic mappings is compactly divergent if and only if each of its subsequences is run-away. ☐

It is a natural to ask whether the necessary conditions given by Propositions 3.1 and 3.3 are sufficient. In Section 5 we give some classes of domains where this is indeed true. But in general this is not so, as we can see using a simple example $\Omega = \mathbb{D}_*$ and $\varphi(z) = \frac{1}{2}z$ (then by Theorems 3.4 or 7.1 the operator C_φ is not hypercyclic, although it satisfies the conditions (c1), (c2), (c3) and (h1), (h2), (h3)).

Theorem 3.4. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Then:*

(1) The operator C_φ is hypercyclic w.r.t. $(n_l)_l$ if and only if φ is injective and for every Ω-convex compact subset $K \subset \Omega$ there exists l such that $K \cap \varphi^{[n_l]}(K) = \varnothing$ and the set $K \cup \varphi^{[n_l]}(K)$ is Ω-convex.

(2) The operator C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$ if and only if φ is injective and for every Ω-convex compact subset $K \subset \Omega$ there exists l_0 such that $K \cap \varphi^{[n_l]}(K) = \varnothing$ and the set $K \cup \varphi^{[n_l]}(K)$ is Ω-convex for any $l \geq l_0$.

Remark 3.5. Note that it suffices to prove the above conditions only for some exhaustion $(K_l)_l$ of Ω with K_l being Ω-convex. This fact is extensively used throughout this paper. Indeed, if $K \subset \Omega$ is an arbitrary compact and Ω-convex set, then taking l_0 such that $K \subset K_{l_0}$ we get that for $l \geq l_0$ the following implication holds: if n_l is such that $K_l \cap \varphi^{[n_l]}(K_l) = \varnothing$ and the set $K_l \cup \varphi^{[n_l]}(K_l)$ is Ω-convex, then the same condition holds with K_l replaced by K. This holds by Lemma 2.5, (3): there exists a function that separates K_l and $\varphi^{[n_l]}(K_l)$, and so it separates K and $\varphi^{[n_l]}(K)$.

Proof of Theorem 3.4. First we prove (1). Necessity. Suppose that C_φ is hypercyclic w.r.t. $(n_l)_l$. Fix K. By Remark 3.2 we get that the set $\varphi^{[n_l]}(K)$ is Ω-convex. Using condition (tt) for $g \equiv 0$, $h \equiv 1$, $\epsilon = \frac{1}{2}$ we get that there are $F \in \mathcal{O}(\Omega, \mathbb{C})$ and $l \in \mathbb{N}$ such that $F(K) \subset \frac{1}{2}\mathbb{D}$ and $F(\varphi^{[n_l]}(K)) \subset 1 + \frac{1}{2}\mathbb{D}$. This implies that K and $\varphi^{[n_l]}(K)$ are disjoint, and by Lemma 2.5 the sum $K \cup \varphi^{[n_l]}(K)$ is Ω-convex.

Sufficiency. We prove that the condition (tt) is satisfied. Fix a compact set $K \subset \Omega$, a number $\epsilon > 0$ and holomorphic functions $g, h : \Omega \to \mathbb{C}$. Take l such that $K \cap \varphi^{[n_l]}(K) = \varnothing$ and the set $I := K \cup \varphi^{[n_l]}(K)$ is Ω-convex. The function $\widetilde{f}$ defined as g in a neighborhood of K and $h \circ \varphi^{[-n_l]}$ in a neighborhood of $\varphi^{[n_l]}(K)$ is well-defined and holomorphic in a neighborhood of an Ω-convex set I, so it can be approximated by functions holomorphic on Ω (see [7, Theorems 4.3.2 and 4.3.4]). This implies that there exists a function $f \in \mathcal{O}(\Omega)$ such that $|f - g| < \epsilon$ on K and $|f - h \circ \varphi^{[-n_l]}| < \epsilon$ on $\varphi^{[n_l]}(K)$.

Observe, that (2) follows immediately from (1). Indeed, the condition in (2) does not hold if and only if there is an Ω-convex subset $K \subset \Omega$ and an increasing sequence $(l_k)_k$ such that for each k the sets K and $\varphi^{[n_{l_k}]}(K)$ are not disjoint or their sum is not Ω-convex. On the other hand, C_φ is not hereditarily hypercyclic w.r.t. $(n_l)_l$ if and only if for some increasing sequence $(l_k)_k$ it is not hypercyclic w.r.t. $(n_{l_k})_k$. In view of (1), these conditions are equivalent. ☐

From the above theorem it follows an interesting observation:

Observation 3.6. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Then the operator C_φ is hypercyclic w.r.t. $(n_l)_l$ if and only if $(n_l)_l$ has a subsequence for which C_φ is hereditarily hypercyclic.*

Proof. Indeed, if C_φ is hypercyclic w.r.t. $(n_l)_l$ and $(K_\mu)_\mu$ is a sequence of Ω-convex compact sets which exhausts Ω, then for each μ there is l_μ such that $K_\mu \cap \varphi^{[n_{l_\mu}]}(K_\mu) = \varnothing$ and the set $K_\mu \cup \varphi^{[n_{l_\mu}]}(K_\mu)$ is Ω-convex. We may assume that the sequence $(l_\mu)_\mu$ is increasing. Then, by Theorem 3.4, the operator C_φ is hereditarily hypercyclic w.r.t. the sequence $(n_{l_\mu})_\mu$ (argue as in Remark 3.5). □

Remark 3.7. In many 'nice' domains Ω there holds a stronger implication than the above: for any $\varphi \in \mathcal{O}(\Omega, \Omega)$ hypercyclicity of C_φ implies its hereditary hypercyclicity. We deal with this topic in Section 6.

Corollary 3.8. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Then:*

(1) The operator C_φ is hypercyclic w.r.t. $(n_l)_l$ if and only if φ is injective, $\varphi(\Omega)$ is a Runge domain w.r.t. Ω and for every Ω-convex compact subset $K \subset \Omega$ there are $l \in \mathbb{N}$ and $F \in \mathcal{O}(\Omega)$ such that the sets $(F(K))\widehat{\ }$ and $\big(F(\varphi^{[n_l]}(K))\big)\widehat{\ }$ are disjoint.

(2) The operator C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$ if and only if φ is injective, $\varphi(\Omega)$ is a Runge domain w.r.t. Ω and for every Ω-convex compact subset $K \subset \Omega$ there exists l_0 such that for any $l \geq l_0$ there is $F \in \mathcal{O}(\Omega, \mathbb{D})$ for which the sets $(F(K))\widehat{\ }$ and $\big(F(\varphi^{[n_l]}(K))\big)\widehat{\ }$ are disjoint.

Proof. Sufficiency in both parts follows immediately from Theorem 3.4. Indeed, if the sets $(F(K))\widehat{\ }$ and $\big(F(\varphi^{[n_l]}(K))\big)\widehat{\ }_\Omega$ are disjoint, then K and $\varphi^{[n_l]}(K)$ are also disjoint and, by Lemma 2.5, their sum is Ω-convex (since $\varphi(\Omega)$ is a Runge domain in Ω, $\varphi^{[n_l]}(K)$ is Ω-convex if K is so).

Necessity. In both parts, using Theorem 3.4 for given K we get suitable l or l_0. Now using Lemma 2.5 for the sets K and $\varphi^{[n_l]}(K)$ we get a required function F. □

Remark 3.9. Observe that from the above corollary it follows that (for an injective φ with $\varphi(\Omega)$ being a Runge domain w.r.t. Ω) to get hypercyclicity of C_φ w.r.t. $(n_l)_l$ it suffices to prove condition (tt) for $g \equiv 0$, $h \equiv 1$ and $\epsilon = \frac{1}{2}$, i.e. to prove that for each compact Ω-convex subset $K \subset \Omega$ there is a number l and a function $f \in \mathcal{O}(\Omega)$ such that

$$|f| < \frac{1}{2} \text{ on } K \text{ and } |f - 1| < \frac{1}{2} \text{ on } \varphi^{[n_l]}(K).$$

Theorem 3.10. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Suppose that φ is injective and that $\varphi(\Omega)$ is a Runge domain w.r.t. Ω. If there exists a point $z_0 \in \Omega$ such that*

$$\lim_{l\to\infty} c_\Omega(z_0, \varphi^{[n_l]}(z_0)) = \infty,$$

then the operator C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$.

Note that the limit condition in the assumptions means that there exists a sequence $(F_l)_l \subset \mathcal{O}(\Omega, \mathbb{D})$ such that $F_l(z_0) = 0$ and $F_l(\varphi^{[n_l]}(z_0)) \to 1$ as $l \to \infty$.

Proof. Since every subsequence of $(n_l)_l$ satisfy the same assumptions, it suffices to prove hypercyclicity. Take $(F_l)_l$ as above. Using the Montel theorem and passing to a subsequence we may assume that $F_l \circ \varphi^{[n_l]} \to G$ and $F_l \to H$ for some functions $G, H \in \mathcal{O}(\Omega)$ with $G(\Omega), H(\Omega) \subset \overline{\mathbb{D}}$. Since $G(z_0) = 1$ and $H(z_0) = 0$, if follows from the maximum principle that $G \equiv 1$ and $H(\Omega) \subset \mathbb{D}$.

Fix a compact Ω-convex subset $K \subset \Omega$. There is an $\alpha \in (0,1)$ so that $H(K) \subset \alpha\mathbb{D}$, so for big l there holds $F_l(K) \subset \alpha\mathbb{D}$. On the other hand, $F_l(\varphi^{[n_l]}(K)) \subset 1 + (1-\alpha)\mathbb{D}$, because $F_l \circ \varphi^{[n_l]} \to 1$. This implies $(F_l(K))\hat{}\cap\big(F_l(\varphi^{[n_l]}(K))\big)\hat{} \subset \alpha\mathbb{D}\cap(1+(1-\alpha)\mathbb{D}) = \varnothing$. Now apply Corollary 3.8. ☐

Remark 3.11. The assumption $\lim_{l\to\infty} c_\Omega(z_0, \varphi^{[n_l]}(z_0)) = \infty$ in Theorem 3.10 is in particular fulfilled in each of the following situations:

(1) If there exists a function $F \in \mathcal{O}(\Omega, \mathbb{D})$ such that $F \circ \varphi^{[n_l]}(z_0) \to 1$ as $l \to \infty$.

(2) If for some point $c \in \partial\Omega$ the sequence $(\varphi^{[n_l]})_l$ converges to a constant c, and there exists a peak function for c and Ω.

Given a sequence $(\varphi_l)_l$ of holomorphic self-maps of a domain $\Omega \subset \mathbb{C}^N$, we say that a sequence $(C_{\varphi_l})_l$ is $\mathcal{B}$-universal if the sequence $(C_{\varphi_l}|_{\mathcal{B}(\Omega)})_l$ is universal as a sequence of mappings from $\mathcal{B}(\Omega)$ to $\mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega) := \mathcal{O}(\Omega, \mathbb{D})$ (endowed with the topology induced from $\mathcal{O}(\Omega)$). In [4, Theorem 3.5 and Corollary 3.7] and [6, Theorem 2.4] the authors proved that, under certain assumptions, $\mathcal{B}$-universality of sequences of composition operators $(C_{\varphi_l})_l$ implies its universality, where φ_l are holomorphic self-maps of Ω. Restricting to the case of iterations, i.e. $\varphi_l = \varphi^{[n_l]}$, and to Ω being pseudoconvex, we give some improvement of their results:

Corollary 3.12. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. Suppose that φ is injective and that $\varphi(\Omega)$ is a Runge domain w.r.t. Ω. If the sequence $(C_\varphi^{[n_l]})_l$ is $\mathcal{B}$-universal, then it is universal, i.e. the operator C_φ is hypercyclic w.r.t. $(n_l)_l$.*

Proof. Let $f \in \mathcal{B}(\Omega)$ be a universal function for $(C_{\varphi_l}|_{\mathcal{B}(\Omega)})_l$, and let $(K_\mu)_\mu$ be an exhaustion of Ω. For every μ there is some number l_μ such that $|f \circ \varphi^{[n_{l_\mu}]} - (1 - \frac{1}{\mu})| < \frac{1}{\mu}$ on K_μ. This gives $f \circ \varphi^{[n_{l_\mu}]} \to 1$ on Ω as $\mu \to \infty$. By Remark 3.11, C_φ is hereditarily hypercyclic w.r.t. $(n_{l_\mu})_\mu$ and hence hypercyclic w.r.t. $(n_l)_l$. ☐

We finish this section with some simple consequence of Theorem 3.4:

Corollary 3.13. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, $\varphi \in \mathcal{O}(\Omega, \Omega)$ and let $(n_l)_l \in \mathbb{N}$ be an increasing sequence. For $M \in \mathbb{N}$ introduce the operator*

$$C_{\varphi,M} : \mathcal{O}(\Omega, \mathbb{C}^M) \ni f \mapsto f \circ \varphi \in \mathcal{O}(\Omega, \mathbb{C}^M).$$

If C_φ is hypercyclic (resp. hereditarily hypercyclic) w.r.t. $(n_l)_l$, then $C_{\varphi,M}$ is hypercyclic (resp. hereditarily hypercyclic) w.r.t. $(n_l)_l$ for every M.

Proof. It is enough to consider the case of hypercyclicity. Obviously φ is injective. We prove that the sequence $(C_{\varphi,M}{}^{[n_l]})_l)$ is topologically transitive (observe that Corollary 2.4 holds in fact also for the operator $C_{\varphi,M}$).

Fix a number $\epsilon > 0$, a compact Ω-convex subset $K \subset \Omega$ and functions $g_1, \dots, g_M$, $h_1, \dots, h_M \in \mathcal{O}(\Omega)$. By Theorem 3.4, there is $l \in \mathbb{N}$ such that the sets K and $\varphi^{[n_l]}(K)$ are disjoint and their sum is Ω-convex. We need to show that there exist functions $f_1, \dots, f_M \in \mathcal{O}(\Omega)$ such that for $j = 1, \dots, M$ there is

$$|f_j - g_j| < \epsilon \text{ on } K \text{ and } |f_j - h_j \circ \varphi^{[-n_l]}| < \epsilon \text{ on } \varphi^{[n_l]}(K).$$

But, as in the proof of Theorem 3.4, this follows from the fact that the functions $\widetilde{f}_j$ defined as g_j in a neighborhood of K and as $h_j \circ \varphi^{[-n_l]}$ in a neighborhood of $\varphi^{[n_l]}(K)$ can be approximated on the Ω-convex set $K \cup \varphi^{[n_l]}(K)$ by functions holomorphic on Ω. $\square$

4 Convergent subsequences

When we resolve Ω-convexity of the set $K \cup \varphi^{[n_l]}(K)$, as it is required in Theorem 3.4, we imagine the set $\varphi^{[n_l]}(K)$ as a compact set lying very close to the boundary of Ω (as it is disjoint with the set K, which is usually big). Thus, it becomes natural to consider the situation when the sets $\varphi^{[n_l]}(K)$ tend to some limit compact set $L(K) \subset \partial\Omega$. There arises a question: what conditions should satisfy $L(K)$ to give hypercyclicity of C_φ w.r.t. $(n_l)_l$? In this section we give a partial answer.

Observe that the assumption that $\varphi^{[n_l]}(K)$ tends to $L(K)$ as $l \to \infty$ implies that the sequence $(\varphi^{[n_l]})_l$ is compactly divergent and bounded on compact subsets of Ω, so by the Montel theorem it has a convergent subsequence (which is also compactly divergent in $\mathcal{O}(\Omega, \Omega)$). Passing to that subsequence we get $\varphi^{[n_l]} \to \widetilde{\varphi}$ on Ω for some holomorphic mapping $\widetilde{\varphi} : \Omega \to \mathbb{C}^N$ with $\widetilde{\varphi}(\Omega) \subset \partial\Omega$, so in fact we can study the situation when $(\varphi^{[n_l]})_l$ has a convergent subsequence which is compactly divergent in $\mathcal{O}(\Omega, \Omega)$.

Note also that the situation considered in this section occurs always when Ω is bounded, since in virtue of the Montel theorem we can always choose a convergent subsequence.

Theorem 4.1. *Let $\Omega, \Omega_0 \subset \mathbb{C}^N$ be pseudoconvex domains, $\Omega \subset \Omega_0$, and let $\varphi : \Omega \to \Omega$ be an injective holomorphic mapping such that the image $\varphi(\Omega)$ is a Runge domain w.r.t. Ω.*

Suppose that for an increasing sequence $(n_l)_l \subset \mathbb{N}$ the subsequence $(\varphi^{[n_l]})_l$ converges to some holomorphic mapping $\widetilde{\varphi} : \Omega \to \mathbb{C}^N$ such that $\widetilde{\varphi}(\Omega) \subset \Omega_0 \cap \partial\Omega$. If for each compact Ω-convex subset $K \subset \Omega$ the set $K \cup (\widetilde{\varphi}(K))\widehat{\ }_{\Omega_0}$ is Ω_0-convex and $(\widetilde{\varphi}(K))\widehat{\ }_{\Omega_0} \subset \partial\Omega$, then C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$.

Remark 4.2. Note that the assumptions of the above theorem imply that the domain Ω is a Runge domain w.r.t. Ω_0. It follows directly from Lemma 2.6.

Proof of Theorem 4.1. We use Theorem 3.4. Fix a compact Ω-convex subset $K \subset \Omega$ and set $L := (\widetilde{\varphi}(K))\widehat{\ }_{\Omega_0}$. Let $U \subset \Omega, V \subset \Omega_0$ be open, disjoint and such that $K \subset U, L \subset V$. Since $U \cup V$ is an open neighborhood of Ω_0-convex set $K \cup L$, we can find an analytic polyhedron (w.r.t. Ω_0) contained in $U \cup V$ and containing $K \cup L$ ([7, Lemma 5.3.7]). Therefore we may assume that $U \cup V = (f_1, \dots, f_\nu)^{-1}(\mathbb{D}^\nu)$ for some functions $f_1, \dots, f_\nu$

holomorphic on Ω_0. For l big enough we have $\varphi^{[n_l]}(K) \subset V \cap \Omega$. Set $I := K \cup \varphi^{[n_l]}(K)$. Remark 4.2 implies that

$$\widehat{I}_\Omega = \widehat{I}_{\Omega_0} \subset U \cup V$$

(the last inclusion follows from the fact that $U \cup V$ is an analytic polyhedrons in Ω_0). Therefore $\widehat{I}_\Omega = \widehat{I}_\Omega \cap \Omega \subset U \cup (V \cap \Omega)$, so by Lemma 2.5 the set I is Ω-convex. $\square$

Theorem 4.3. *Let $\Omega_0 \subset \mathbb{C}^N$ be a pseudoconvex domain, $c \in \mathbb{R}$, $u \in PSH(\Omega_0)$ and let Ω be a connected component of the set $\{u < c\}$. Suppose that near a point $z_0 \in \Omega_0 \cap \partial\Omega$ the function u is strictly plurisubharmonic.*

Let $\varphi : \Omega \to \Omega$ be an injective holomorphic mapping with $\varphi(\Omega)$ being a Runge domain w.r.t. Ω. If for an increasing sequence $(n_l)_l \in \mathbb{N}$ the subsequence $(\varphi^{[n_l]})_l$ is convergent to a holomorphic mapping $\widetilde{\varphi} : \Omega \to \mathbb{C}^N$ such that $z_0 \in \widetilde{\varphi}(\Omega)$, then C_φ is hereditarily hypercyclic w.r.t. $(n_l)_l$.

This theorem works in the case when the image of the limit function $\widetilde{\varphi}$ contains a point of strict convexity of Ω, i.e. a point $z_0 \in \partial\Omega$ for which there exists a strictly plurisubharmonic local defining function for Ω.

Proof. Let $z_1 \in \Omega$ be such that $\widetilde{\varphi}(z_1) = z_0$. The function $u \circ \widetilde{\varphi}$ is plurisubharmonic and bounded from above by c. In the point z_1 it achieves its local maximum, so it is constant and equal to c near z_1. Therefore for each z near z_1 and $X \in \mathbb{C}^N$ we have

$$0 = \frac{\partial^2(u \circ \widetilde{\varphi})}{\partial\lambda\partial\overline{\lambda}}\bigg|_{\lambda=0} (z + \lambda X) = \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(\widetilde{\varphi}(z))\, Y_j \overline{Y_k},$$

where $Y = \widetilde{\varphi}'(z)(X)$. But the sum on the right side is positive if $Y \neq 0$, because u is strictly plurisubharmonic near the boundary point $\widetilde{\varphi}(z)$. This implies $\widetilde{\varphi}'(z)(X) = 0$ for every X, so $\widetilde{\varphi}$ is constant and equal to z_0. In particular, $\widetilde{\varphi}(\Omega) \subset \Omega_0 \cap \partial\Omega$.

Observe that Ω is a Runge domain w.r.t. Ω_0. Indeed, if $K \subset \Omega$ is compact, connected and Ω-convex, then

$$\widehat{K}_{\Omega_0} = \widehat{K}_{PSH(\Omega_0)} \subset\subset \{u < c\},$$

so $\widehat{K}_{\Omega_0} \subset \Omega$. Here $\widehat{K}_{PSH(\Omega_0)}$ denotes the plurisubharmonic hull of K w.r.t. Ω_0 ([7, Definition 2.6.6]); it is equal to $\widehat{K}_{\Omega_0}$ when Ω_0 is pseudoconvex ([7, Theorem 4.3.4]).

In virtue of Theorem 4.1 it suffices to prove that the set $K \cup \{z_0\}$ is Ω_0-convex if $K \subset \Omega$ is compact and Ω-convex. Such a set K is Ω_0-convex, so there is $f \in \mathcal{O}(\Omega_0)$ with $|f(z_0)| > \sup_{z \in K} |f(z)|$. Now, Lemma 2.5 for $L = \{z_0\}$ does the job, because f satisfies the condition (3). $\square$

5 Simply characterization of hypercyclicity

The general characterization of hypercyclicity given by Theorem 3.4 requires resolving Ω-convexity of the sets $K \cup \varphi^{[n]}(K)$, while characterization in $\mathbb{C}$, given by [6, Theorem 3.21] (see also Theorem 7.2 in this paper), requires only satisfying the necessary conditions (c1), (c2), (c3), which is much simpler. Hence there arises a natural question about domains in $\mathbb{C}^N$ in which that necessary conditions are also sufficient for hypercyclicity. A simple example $\Omega := \mathbb{D} \setminus \{0\}$ with the mapping $\varphi(z) := \frac{1}{2}z$ shows that there are domains where this is not true. In this section we present some natural classes of domains which supports this 'simply characterization'.

Definition 5.1. Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain. By saying that '*(c1), (c2), (c3) are sufficient for hypercyclicity in* Ω' (resp. '*(h1), (h2), (h3) are sufficient for hereditary hypercyclicity in* Ω') we mean that the operator C_φ is hypercyclic w.r.t. $(n_l)_l$ (resp. hereditarily hypercyclic w.r.t. $(n_l)_l$) for every mapping $\varphi \in \mathcal{O}(\Omega, \Omega)$ and every increasing sequence $(n_l)_l$ satisfying the conditions (c1), (c2), (c3) (resp. (h1), (h2), (h3)).

Let us begin with a general fact:

Proposition 5.2. *Let $\Omega \in \mathbb{C}^N$ be a pseudoconvex domain. Then (c1), (c2), (c3) are sufficient for hypercyclicity in Ω if and only if (h1), (h2), (h3) are sufficient for hereditary hypercyclicity in Ω.*

Since the notions defined above are equivalent, we formulate the following definition:

Definition 5.3. A pseudoconvex domain $\Omega \in \mathbb{C}^N$ is called *hypercyclic* if the conditions (c1), (c2), (c3) are sufficient for hypercyclicity in Ω.

Proof of Proposition 5.2. Suppose (c1), (c2), (c3) are sufficient for hypercyclicity in Ω and choose φ and $(n_l)_l$ satisfying (h1), (h2), (h3). Then every subsequence of $(n_l)_l$ satisfies (c1), (c2), (c3), so every subsequence of $({C_\varphi}^{[n_l]})_l$ is universal.

Conversely, suppose that (h1), (h2), (h3) are sufficient for hereditary hypercyclicity in Ω and choose φ and $(n_l)_l$ satisfying (c1), (c2), (c3). Since the sequence $(\varphi^{[n_l]})_l$ is run-away, it has a compactly divergent subsequence. This subsequence satisfies (h1), (h2), (h3), so $({C_\varphi}^{[n_l]})_l$ is universal, because it has a hereditarily universal subsequence. ☐

Theorem 5.4. *If $\Omega \subset \mathbb{C}^N$ is a pseudoconvex domain for which there exists a point $z_0 \in \Omega$ so that*

$$\lim_{\Omega \ni z \to \infty_\Omega} c_\Omega(z, z_0) = \infty,$$

then Ω is hypercyclic.

Note that the assumption $\lim_{\Omega \ni z \to \infty_\Omega} c_\Omega(z, z_0) = \infty$ means that all the balls w.r.t. Carathéodory pseudodistance are relatively compact in Ω. If it holds for some z_0, then it holds for every $z_0 \in \Omega$ (this is an easy consequence of the triangle inequality).

Proof. Choose φ and $(n_l)_l$ satisfying (c1), (c2), (c3). Since $(\varphi^{[n_l]})_l$ is run-away, by passing to a subsequence we may assume that it is compactly divergent in $\mathcal{O}(\Omega, \Omega)$. We have $\varphi^{[n_l]}(z_0) \to \infty_\Omega$, so $c_\Omega(z_0, \varphi^{[n_l]}(z_0)) \to \infty$. Now Theorem 3.10 does the job. ☐

Although most of the classes of domains which we consider below are bounded, note that this assumption is not required in the theorem above.

Corollary 5.5. *The following domains satisfy the assumptions of Theorem 5.4 and hence are hypercyclic:*

(1) A bounded pseudoconvex domain $\Omega \subset \mathbb{C}^N$ such that for each point $a \in \partial\Omega$ there is a function $f_a \in \mathcal{O}(\Omega, \mathbb{D})$ such that $f_a(z) \to 1$ as $z \to a, z \in \Omega$.

(2) A bounded strictly pseudoconvex domain in $\mathbb{C}^N$.

(3) A bounded convex domain in $\mathbb{C}^N$.

(4) A relatively compact analytic polyhedron w.r.t. a pseudoconvex domain $\Omega_0 \Subset \mathbb{C}^N$, i.e. a domain Ω which is relatively compact in Ω_0 and is a connected component of the set $(f_1, \dots, f_m)^{-1}(\mathbb{D}^m)$ for some functions $f_1, \dots, f_m \in \mathcal{O}(\Omega_0)$.

For hypercyclic domains in $\mathbb{C}$ see Theorem 7.2.

Proof. (1): Suppose that for some sequence $(z_n)_n \in \Omega$, $z_n \to \infty_\Omega$ and some constant there is $c_\Omega(z_0, z_n) < M$. Since Ω is bounded, we have $z_n \to \partial\Omega$ and (passing to a subsequence) we may assume that $z_n \to a$ for some boundary point a. Therefore $f_a(z_n) \to 1$, so $c_\Omega(z_0, z_n) \to \infty$; a contradiction.

(2): It follows from [9, Theorem 10.2.1].

(3): This is a consequence of (1). Indeed, if $a \in \partial\Omega$, then there is a real linear functional $L : \mathbb{C}^N \to \mathbb{R}$ such that $L(a) = 0$ and $L < 0$ on Ω. We have $L = \operatorname{Re} \widetilde{L}$ for some complex linear functional $\widetilde{L}$. Defining $f(z) := e^{\widetilde{L}(z) - i \operatorname{Im} \widetilde{L}(a)}$ we obtain a peak function for a.

(4): This is also an immediate consequence of (1). $\square$

6 Hereditary hypercyclicity of C_φ

In this section we deal a bit more with hereditary hypercyclicity. As it was said in Observation 3.6, hypercyclicity of C_φ is equivalent to its hereditary hypercyclicity w.r.t. some increasing sequence $(n_l)_l \in \mathbb{N}$. But, as we shall see in this section, there are domains in which for every mapping $\varphi \in \mathcal{O}(\Omega, \Omega)$ hypercyclicity of C_φ gives its hereditary hypercyclicity, what is the strongest of conditions (p1) - (p4).

First, let us recall the following fact (see [1, Theorem 2.4.3]):

Theorem 6.1. *Let $\Omega \subset \mathbb{C}^N$ be a taut domain and let $\varphi \in \mathcal{O}(\Omega, \Omega)$. If the sequence $(\varphi^{[n]})_n$ is not compactly divergent in $\mathcal{O}(\Omega, \Omega)$, then it is relatively compact in $\mathcal{O}(\Omega, \Omega)$.*

We say that a domain $\Omega \Subset \mathbb{C}^N$ is taut if every sequence $(f_n)_n \in \mathcal{O}(\mathbb{D}, \Omega)$ is compactly divergent in $\mathcal{O}(\mathbb{D}, \Omega)$ or has a subsequence convergent to an element of $\mathcal{O}(\mathbb{D}, \Omega)$; it is well known that every taut domain in $\mathbb{C}^N$ is pseudoconvex. The reader can find more informations in [9].

We prove the following:

Theorem 6.2. *Let $\Omega \subset \mathbb{C}^N$ be a hypercyclic taut domain and let $\varphi : \Omega \to \Omega$ be a holomorphic mapping. If the operator C_φ is hypercyclic, then it is hereditarily hypercyclic.*

Note that tautness does not imply hypercyclicity of a domain, and vice versa. The examples are simply: $\mathbb{D}_*$ is taut, but not hypercyclic, while $\mathbb{C}$ is hypercyclic, but not taut (see Theorem 7.2).

Proof. Hypercyclicity of C_φ implies that φ satisfies the conditions (c1), (c2), (c3) with $n_l = l$. The last condition implies that $(\varphi^{[n]})_n$ cannot be relatively compact in $\mathcal{O}(\Omega, \Omega)$, because it has a compactly divergent subsequence. Therefore by Theorem 6.1 the sequence $(\varphi^{[n]})_n$ is compactly divergent, so φ satisfies the conditions (h1), (h2), (h3) with $n_l = l$. Then, as Ω is hypercyclic, the operator C_φ must be hereditarily hypercyclic. $\square$

Combining Theorems 5.4 and 6.2 we get:

Theorem 6.3. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain for which there exists a point $z_0 \in \Omega$ so that*
$$\lim_{\Omega \ni z \to \infty_\Omega} c_\Omega(z, z_0) = \infty.$$
Then Ω is taut and hypercyclic.

In particular, if $\varphi \in \mathcal{O}(\Omega, \Omega)$ is such that C_φ if hypercyclic, then C_φ is hereditarily hypercyclic.

Remark 6.4. By Corollary 5.5, for all the domains listed there hypercyclicity of C_φ implies its hereditary hypercyclicity. For domains in $\mathbb{C}$ having this property see Theorem 7.2.

Example 8.5 shows that there are taut domains which does not satisfy the assumptions of Theorem 6.3, but satisfy its conclusion.

Proof of Theorem 6.3. In virtue of Theorems 5.4 and 6.2 it suffices to prove that Ω is taut. Actually, this is a well-known fact, but we could not find it in the literature in that form, so for the reader's convenience we present a sketch of proof.

Fix a sequence $(f_n)_n \subset \mathcal{O}(\mathbb{D}, \Omega)$ and suppose that it is not compactly divergent. Passing to a subsequence (during the proof we do it few times) we may assume that there are compact sets $K \subset \mathbb{D}$ and $M \subset \Omega$ such that $f_n(K) \cap M \neq \varnothing$ for every n.

We show that for each compact subset $L \subset \mathbb{D}$ the set $I_L := \bigcup_{n=1}^\infty f_n(L)$ is relatively compact in Ω. Suppose that for some L it is not. Hence there is a sequence $(w_k)_k \subset I_L$ such that $w_k \to \infty_\Omega$ as $k \to \infty$. There is $w_k = f_{n_k}(z_k)$ for some sequences $(z_k)_k \subset L$ and $(n_k)_k \subset \mathbb{N}$. Since for each m the sum $\bigcup_{n=1}^m f_n(L)$ is relatively compact in Ω, the set $\{n_k : k \in \mathbb{N}\}$ is infinite. Therefore we may assume that the sequence $(n_k)_k$ is increasing and, in addition, that z_k tends to some $z_0 \in L$.

For every k there is a function $F_k \in \mathcal{O}(\Omega, \mathbb{D})$ such that $F_k(z_0) = 0$ and
$$F_k \circ f_{n_k}(z_k) = F_k(w_k) = c^*_\Omega(z_0, w_k).$$
By the Montel theorem, we may assume that $F_k \to G$ on Ω and $F_k \circ f_{n_k} \to H$ on $\mathbb{D}$ for some functions $G \in \mathcal{O}(\Omega), H \in \mathcal{O}(\mathbb{D})$, with $G(\Omega), H(\mathbb{D}) \subset \overline{\mathbb{D}}$. We have $G(z_0) = 0$ and
$$H(z_0) = \lim_{k\to\infty} F_k \circ f_{n_k}(z_k) = \lim_{k\to\infty} c^*_\Omega(z_0, w_k) = 1.$$
In view of the maximum principle, $H \equiv 1$ and $G(\Omega) \subset \mathbb{D}$. Since $f_{n_k}(K) \cap M \neq \varnothing$ for each k, there is
$$\min_K |F_k \circ f_{n_k}| = \min_{f_{n_k}(K)} |F_k| \leq \min_{f_{n_k}(K)\cap M} |F_k| \leq \max_{f_{n_k}(K)\cap M} |F_k| \leq \max_M |F_k|.$$
The left side tends to 1, while the right side tends to $\max_M |G|$, which is less than 1; a contradiction.

The set I_L is relatively compact in Ω for every compact set L, so the sequence $(f_n)_n$ is locally uniformly bounded on $\mathbb{D}$. By the Montel theorem is has a subsequence convergent to a map $\widetilde{f} \in \mathcal{O}(\mathbb{D}, \mathbb{C}^N)$. The inclusion $\widetilde{f}(\mathbb{D}) \subset \Omega$ follows again from the fact that $I_L \subset\subset \Omega$. $\square$

Unfortunately, we were not able neither to construct an example of Ω and φ for which C_φ is hypercyclic but non-hereditarily hypercyclic, nor to prove that such Ω and φ do not exist. What we can show is only:

Proposition 6.5. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain and $\varphi \in \mathcal{O}(\Omega, \Omega)$.*

(1) If C_φ is hypercyclic, then it is hypercyclic with respect to each increasing arithmetic sequence of natural numbers.

(2) If C_φ is hereditarily hypercyclic with respect to some increasing arithmetic sequence of natural numbers, then it is hereditarily hypercyclic.

Proof. (1): Choose a sequence of the form $(l\mu + \nu)_l$, where $\mu \in \mathbb{N}$, $\mu > 0$ and $\nu \in \{0, \dots, \mu - 1\}$. The fact that C_φ if hypercyclic w.r.t $(l\mu)_l$ follows from [5, Theorem 8] (this theorem states even more: the sequences $({C_\varphi}^{[l]})_l$ and $({C_\varphi}^{[l\mu]})_l$ have the same set of universal elements). Let f be a universal function for $({C_\varphi}^{[l\mu]})_l$.

Take $h \in \mathcal{O}(\Omega)$. The function $h \circ \varphi^{[-\nu]}$ is holomorphic on $\varphi^{[\nu]}(\Omega)$ which is a Runge domain w.r.t. Ω. Therefore it can be approximated by elements of $\mathcal{O}(\Omega)$, so for some increasing sequence $(l_k)_k \subset \mathbb{N}$ there is $f \circ \varphi^{[l_k\mu]} \to h \circ \varphi^{[-\nu]}$ on $\varphi^{[\nu]}(\Omega)$, as $k \to \infty$. Hence $f \circ \varphi^{[l_k\mu+\nu]} \to h$ on Ω.

(2): Assume that C_φ is hereditarily hypercyclic w.r.t. a sequence $(l\mu+\nu)_l$, where μ, ν are as above. It suffices to show that for each $r \in \{0, \dots, \mu - 1\} \setminus \{\nu\}$ the operator C_φ is hereditarily hypercyclic w.r.t. $(l\mu + r)_l$.

Fix r and an increasing sequence $(l_k)_k \subset \mathbb{N}$. We have $\mu + r - \nu > 0$. As previously, given $h \in \mathcal{O}(\Omega)$ we approximate the function $h \circ \varphi^{[\nu-r-\mu]}$ on $\varphi^{[\mu+r-\nu]}(\Omega)$ by functions from $\mathcal{O}(\Omega)$. The sequence $({C_\varphi}^{[(l_k-1)\mu+\nu]})_k$ is universal, so there is an increasing sequence $(k_m)_m \subset \mathbb{N}$ such that

$$f \circ \varphi^{[(l_{k_m}-1)\mu+\nu]} \to h \circ \varphi^{[\nu-r-\mu]} \text{ on } \varphi^{[\mu+r-\nu]}(\Omega) \text{ as } m \to \infty.$$

This implies $f \circ \varphi^{[l_{k_m}\mu+r]} \to h$ on Ω, because the mapping $\varphi^{[\mu+r-\nu]} : \Omega \to \varphi^{[\mu+r-\nu]}(\Omega)$ is a biholomorphism. $\square$

7 One-dimensional case

In [6, Theorems 3.19 and 3.21] Grosse-Erdmann and Mortini gave a complete characterization of hypercyclicity in one-dimensional case, which after a slight reformulation takes the following form:

Theorem 7.1. *Let $(\varphi_l)_l$ be a sequence of injective holomorphic self-maps of a domain $\Omega \subset \mathbb{C}$. Then:*

(1) If Ω is simply connected, then $(C_{\varphi_l})_l$ is universal if and only if $(\varphi_l)_l$ is run-away.

(2) If Ω is finitely connected but not simply connected, then $(C_{\varphi_l})_l$ is never universal.

(3) If Ω is infinitely connected, then $(C_{\varphi_l})_l$ is universal if and only if for every compact Ω-convex subset $K \subset \Omega$ and for every l_0 there is $l \geq l_0$ such that $\varphi_l(K)$ is Ω-convex and $\varphi_l(K) \cap K = \varnothing$.

Although Grosse-Erdmann and Mortini defined Ω-convexity in different way (they said that a compact subset $K \subset \Omega$ of a domain $\Omega \subset \mathbb{C}$ is Ω-convex if every hole of K contains a point of $\mathbb{C} \setminus \Omega$), our definition agrees with their in dimension one. Here by hole of K we mean a bounded connected component of $\mathbb{C} \setminus K$. Equivalence of both definitions follows from [7, Theorems 1.3.1 and 1.3.3].

As a corollary from our considerations and from the above theorem applied to mappings $\varphi_l := \varphi^{[n_l]}$ we obtain:

Theorem 7.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected or an infinitely connected domain. Then:*

(1) Ω is hypercyclic.

(2) If for a mapping $\varphi \in \mathcal{O}(\Omega, \Omega)$ the operator C_φ is hypercyclic, then it is hereditarily hypercyclic.

Proof. Let Ω be simply connected or infinitely connected. Part (1) follows directly from Theorem 7.1. We prove (2). It is known that a domain $\Omega \subset \mathbb{C}$ is taut if and only if the set $\mathbb{C} \setminus \Omega$ has at least two points (see [9, Remark 3.2.3 (d)]), so if $\Omega \neq \mathbb{C}$, then Ω is taut and Theorem 6.2 does the job.

It remains to consider the case $\Omega = \mathbb{C}$. Fix φ for which C_φ is hypercyclic. By the Picard theorem the set $\mathbb{C} \setminus \varphi(\mathbb{C})$ contains at most one point. But since φ is a homeomorphism on its image, there must be $\varphi(\mathbb{C}) = \mathbb{C}$ and so φ is an automorphism of $\mathbb{C}$. Therefore φ is an affine endomorphism. Now it suffices to use [2, Theorem 3.1]: it says (in particular) that for φ being an affine endomorphism of $\mathbb{C}^N$, the composition operator $C_\varphi : \mathcal{O}(\mathbb{C}^N) \to \mathcal{O}(\mathbb{C}^N)$ is hypercyclic if and only if it is hereditarily hypercyclic. $\square$

As an interesting observation (but not connected with the topic of this paper), it follows from Corollary 5.5 and Theorem 7.2 a well-known fact that:

Remark 7.3. No holomorphic self-mapping of an annulus in $\mathbb{C}$ satisfy all the conditions (c1), (c2) and (c3).

8 Examples

Let us begin with the following observation:

Proposition 8.1. *Let $a \in \mathbb{C}^{N_2}$ and let $D \subset \mathbb{C}^{N_1}, \Omega \subset \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ be pseudoconvex domains such that*

$$D \times \{a\} \subset \Omega \subset D \times \mathbb{C}^{N_2}.$$

Let $\varphi : \Omega \to \Omega$ be a holomorphic mapping of the form

$$\varphi(z, w) = (\sigma(z), \psi(z, w)),\ (z, w) \in \Omega,$$

where $\sigma : D \to D$ and $\psi : \Omega \to \mathbb{C}^{N_2}$.

If φ is injective, the image $\varphi(\Omega)$ is a Runge domain w.r.t. Ω and the operator C_σ is hypercyclic (resp. hereditarily hypercyclic) w.r.t. an increasing sequence $(n_l)_l \subset \mathbb{N}$, then C_φ is also hypercyclic (resp. hereditarily hypercyclic) w.r.t. $(n_l)_l$.

Proof. It is enough to consider the case of hypercyclicity. Fix $K \subset \Omega$ compact and Ω-convex, and set $L := (\pi_{\mathbb{C}^{N_2}}(K))\widehat{\ }_D$. By Corollary 3.8, there are $l \in \mathbb{N}$ and $f \in \mathcal{O}(D)$ such that

$$\big(f(\sigma^{[n_l]}(L))\big)\widehat{\ } \cap (f(L))\widehat{\ } = \varnothing.$$

Define $g(z, w) := f(z)$ for $(z, w) \in \Omega$. We have $\varphi^{[n_l]}(K) \subset \sigma^{[n_l]}(L) \times \mathbb{C}^{N_2}$, so

$$\big(g(\varphi^{[n_l]}K)\big)\widehat{\ } \cap (g(K))\widehat{\ } \subset \big(f(\sigma^{[n_l]}(L))\big)\widehat{\ } \cap (f(L))\widehat{\ } = \varnothing,$$

so C_φ is hypercyclic w.r.t. $(n_l)_l$ (use Corollary 3.8). □

We get an immediate corollary:

Corollary 8.2. *For any $j = 1, \ldots, m$, let $\Omega_j \subset \mathbb{C}^{N_j}$ be a pseudoconvex domain and let $\varphi_j \in \mathcal{O}(\Omega_j, \Omega_j)$. Suppose that for every j the mapping φ_j is injective and its image is a Runge domain w.r.t. Ω_j. Define $\Omega := \Omega_1 \times \ldots \times \Omega_m$ and $\varphi := \varphi_1 \times \ldots \times \varphi_m$.*

If there exists j_0 such that the operator $C_{\varphi_{j_0}}$ is hypercyclic w.r.t. an increasing sequence $(n_l)_l \subset \mathbb{N}$, then the operator C_φ is hypercyclic w.r.t. $(n_l)_l$.

Example 8.3. Put $\Omega = \mathbb{D} \times \mathbb{D}_*$. Then, although $\mathbb{D}_*$ does not have any self-mappings with hypercyclic composition operator, the domain Ω has many of them, e.g. every mapping of the form $\varphi(z, w) = (\psi(z), w)$ for $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ such that C_ψ is hypercyclic.

Example 8.4. Consider the Hartogs domain given by

$$\Omega := \{(z, \zeta) \in D \times \mathbb{C} : |\zeta| e^{u(z)} < 1\},$$

where $D \subset \mathbb{C}^N$ is pseudoconvex and $u : D \to [-\infty, \infty)$ is plurisubharmonic, $u \not\equiv -\infty$. It is well-known that such a domain Ω is pseudoconvex. Consider the following situation, which is in a certain sense opposite to that in Proposition 8.1: let φ be of the form

$$\varphi(z, \zeta) = (z, \psi(z, \zeta)),$$

where $\psi \in \mathcal{O}(\Omega, \mathbb{C})$.

Suppose that C_φ is hypercyclic. Then:

(1) The function u is equal to $\log |G|$ for some $G \in \mathcal{O}(D, \mathbb{C}_*)$, and hence the domain Ω is biholomorphic to $D \times \mathbb{D}$.

(2) The operator C_φ is hereditarily hypercyclic.

Put $\psi_z := \psi(z, \cdot)$, $\Omega_z = \{\zeta \in \mathbb{C} : |\zeta| < e^{-u(z)}\}$ for $z \in D$. Then $\psi_z \in \mathcal{O}(\Omega_z, \Omega_z)$ and Ω_z is a disc when $u(z) \neq -\infty$, and the whole $\mathbb{C}$ in the opposite case. We have

$$\varphi^{[n]}(z, \zeta) = (z, {\psi_z}^{[n]}(\zeta))$$

for each $n \in \mathbb{N}$. Since φ is injective and the sequence $(\varphi^{[n]})_n$ is run-away, for each $z \in D$ the mapping ψ_z is injective and the sequence $({\psi_z}^{[n]})_n$ is run-away. Therefore, as Ω_z is simply connected, in view of Theorems 7.1 and 7.2, the operator C_{ψ_z} is hereditarily hypercyclic for every z.

We prove (1). As the sequence $(\varphi^{[n]})_n$ is run-away, it has a compactly divergent subsequence, say $(\varphi^{[\mu_l]})_l$. Set

$$\tau_l(z) := {\psi_z}^{[\mu_l]}(0) \text{ for } z \in D.$$

The sequence $(id_D, \tau_l)_l$ is compactly divergent in $\mathcal{O}(D, \Omega)$, because $\varphi^{[\mu_l]}(0, z) = (z, \tau_l(z))$.

Let $W \subset\subset D$ be a domain. As u is upper semi continuous, it is bounded from above on $\overline{W}$, so there is an $\epsilon > 0$ such that $\overline{W} \times \epsilon\overline{\mathbb{D}} \subset \Omega$. As $(id_D, \tau_l)_l$ is compactly divergent, we get that for l greater than some l_0 the functions τ_l are zero-free on $\overline{W}$ and the sequence $(\frac{1}{\tau_l})_{l \geq l_0}$ is uniformly bounded on $\overline{W}$.

Using the above conclusion and the Montel theorem, we shall prove that the sequence $(\frac{1}{\tau_l})_l$ has a subsequence convergent locally uniformly on D to a function $H \in \mathcal{O}(D)$. Indeed, let $(W_p)_p$ be a sequence of domains such that $W_p \subset\subset W_{p+1}$ and $\bigcup_p W_p = D$. We apply the Montel theorem. There is an increasing sequence $(l_p^1)_p \subset \mathbb{N}$ such that $\frac{1}{\tau_{l_p^1}}$ converges to a function $H_1 \in \mathcal{O}(W_1)$ locally uniformly on W_1. Next, there is $(l_p^2)_p \subset (l_p^1)_p$ such that $\frac{1}{\tau_{l_p^2}}$ converges to $H_2 \in \mathcal{O}(W_2)$ on W_2. Doing so, for every q we get a sequence $(l_p^q)_p$ such that $\frac{1}{\tau_{l_p^q}}$ converges to $H_q \in \mathcal{O}(W_q)$ on W_q, and $(l_p^q)_p \subset (l_p^{q-1})_p$. We have $H_{q_1} = H_{q_2}$ on $W_{q_1} \cap W_{q_2}$, and hence we may define a function $H \in \mathcal{O}(D)$ as $H(z) = H_q(z)$ for $z \in W_q$. For each q there is l_q such that $|H - \frac{1}{\tau_{l_q}}| < \frac{1}{q}$ on W_q. We may assume that the sequence $(l_q)_q$ is increasing. Then the sequence $\frac{1}{\tau_{l_q}}$ converges to H locally uniformly on D.

As it was said, the sequence $(id_D, \tau_l)_l$ is compactly divergent in $\mathcal{O}(D, \Omega)$. Therefore $|\tau_l(z)| \to e^{-u(z)}$ as $l \to \infty$ for each $z \in D$, and hence

$$|H(z)| = e^{u(z)} \text{ for } z \in D.$$

Observe that H is zero-free on D. Indeed, as $u \not\equiv -\infty$, there is $H \not\equiv 0$. If $H(z) = 0$ for some z, then by the Hurwitz theorem $\frac{1}{\tau_{l_p}}$ has a zero near z for p big enough, what is a contradiction.

The mapping

$$\Omega \ni (z, \zeta) \mapsto (z, \zeta H(z)) \in D \times \mathbb{D}$$

is a required biholomorphism between Ω and $D \times \mathbb{D}$.

We prove (2). Via the above biholomorphic equivalence we may assume that $\Omega = D \times \mathbb{D}$. Fix an increasing sequence $(n_l)_l \in \mathbb{N}$. The sequence $(\varphi^{[n_l]})_{n_l}$ is locally uniformly bounded on Ω, so passing to a subsequence we get $\varphi^{[n_l]} \to \widetilde{\varphi}$. There is $\widetilde{\varphi}(z, \zeta) = (z, \widetilde{\psi}(z, \zeta))$ for some $\widetilde{\psi} \in \mathcal{O}(\Omega, \mathbb{C})$ such that $\widetilde{\psi}(\Omega) \subset \partial\mathbb{D}$. Therefore $\widetilde{\psi}$ is equal to some constant $c \in \partial\mathbb{D}$. Putting $f(z, \zeta) := \frac{1}{c}\zeta$ allows us to apply Remark 3.11. The part (2) is proved.

Note that in general there exist mappings φ of the form considered above for which the operator C_φ is hypercyclic. Indeed, put $D := \mathbb{D}$. Via conformal equivalence we may replace the set $D \times \mathbb{D}$ with $\Omega := H_+ \times H_+$, where H_+ is the right half-plane in $\mathbb{C}$, i.e. $H_+ := \{\mathrm{Re} > 0\}$. Now set $\Phi(z, w) := (z, w + z)$ and $\varphi := \Phi|_\Omega$. It is clear that $\varphi(\Omega) \subset \Omega$, φ is an injection and the sequence $(\varphi^{[n]})_n$ is compactly divergent in $\mathcal{O}(\Omega, \Omega)$. The mapping Φ is an automorphism of $\mathbb{C}^2$ and Ω is a Runge domain in $\mathbb{C}^2$, so the image $\Phi(\Omega)$ is also a Runge domain in $\mathbb{C}^2$ and hence in Ω. Therefore the conditions (h1), (h2), (h3) are satisfied for φ. By Theorem 6.3, the domain Ω is hypercyclic and the operator C_φ is hereditarily hypercyclic.

Example 8.5. Let $\Omega_0 \subset \mathbb{C}$ be an infinitely connected domain and let $a \in \Omega_0$. Define $\Omega := \Omega_0 \setminus \{a\}$. Then by Theorem 7.2 the domain Ω satisfies the conclusion of Theorem 6.3, but it does not satisfy the assumption that $\lim_{\Omega \ni z \to \infty_\Omega} c_\Omega(z, z_0) = \infty$. This follows from the classical Riemann extension theorem: there is $c_\Omega = c_{\Omega_0}$ on $\Omega, \times \Omega$ and $c_\Omega(z_0, z) \to c_{\Omega_0}(z_0, a) < \infty$ as $z \to a$, for any z_0.

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